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Kac-Moody symmetries of ten-dimensional non-maximal supergravity theories

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Abstract: A description of the bosonic sector of ten-dimensional $N = 1$ supergravity as a non-linear realisation is given. We show that if a suitable extension of this theory were invariant under a Kac-Moody algebra, then this algebra would have to contain a rank eleven Kac-Moody algebra, that can be identified to be a particular real form of very-extended D_8. We also describe the extension of $N = 1$ supergravity coupled to an abelian vector gauge field as a non-linear realisation, and find the Kac-Moody algebra governing the symmetries of this theory to be very-extended B_8. Finally, we discuss the related points for the $N = 1$ supergravity coupled to an arbitrary number of abelian vector gauge fields.

Keywords: Space-Time Symmetries, Gauge Symmetry, Global Symmetries, Supergravity Models.

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1. Introduction

There are no supergravity theories in space-time dimensions greater than eleven, but there exists a unique supergravity theory in eleven dimensions and two supergravity theories in ten dimensions which differ according to whether the two 16 component Majorana-Weyl spinors are of same or opposite chirality [1]. The corresponding ten dimensional supergravity theories are called IIA [2] and IIB [3] respectively and form the low energy effective actions of the string theories with the same name. There exists one other supergravity theory in ten dimensions which possesses a supersymmetry with a Majorana-Weyl spinor that can be coupled to Yang-Mills supermultiplets [4]. The quantum version of this theory is plagued with anomalies, but these can be canceled by introducing Yang-Mills multiplets whose gauge groups are SO(32) or $E_8 \otimes E_8$ [5]. These two anomaly-free theories are the low energy effective actions for the SO(32) and heterotic superstring theories [6].

It is a consequence of supersymmetry that the scalars in supergravity multiplets belong to non-linear realisations; the first such example was discovered in [7]. One of the most celebrated examples concerns the four dimensional maximal supergravity where the scalars belong to a non-linear realisation of E_7 [8]. In [9] the coset construction was extended to include the gauge fields of supergravity theories. This method used generators that are inert under Lorentz transformations and, as such, it is difficult to extend this method to include both gravity and fermions. The eleven dimensional supergravity theory does not possess any scalars and it was widely believed that the symmetry algebras, such as E_7, found on dimensional reduction were not present in this theory. However, it was found that the eleven dimensional supergravity theory could be formulated as a non-linear realisation [10]. The infinite dimensional algebra involved in this construction was the closure of a finite dimensional algebra, denoted G_{11}, with the conformal algebra in eleven dimensions. The non-linear realisation was carried out by ensuring that the equations of motion were invariant under both finite dimensional algebras, taking into account that some of their generators were in common. The algebra G_{11} involved the space-time translations together with an algebra $\hat{G}_{11}$ which contained A_{10} and the Borel subalgebra of E_7 as subalgebras. The algebra $\hat{G}_{11}$ was not a Kac-Moody algebra, however, it was conjectured [11] that the theory could be extended so that the algebra $\hat{G}_{11}$ was promoted to a Kac-Moody algebra. It was shown that this symmetry would have to contain a certain rank eleven Kac-Moody algebra denoted E_{11} [11].

The bosonic sector of IIB supergravity theory can also be described as a non-linear realisation [12] and the corresponding Kac-Moody algebra is E_{11}. The massive extension of IIA was formulated as a non-linear realisation in [13].

In this paper we construct $N = 1$ pure supergravity as a non-linear realisation, and identify the underlying Kac-Moody algebra to be D_8. We then extend the construction to the case when the theory includes a number of abelian vector field and find underlying the Kac-Moody algebra. Finally, we discuss how one might extend this result to the case when the vector fields are part of a non-abelian gauge symmetry.

2. Pure $N = 1$ supergravity

The lagrangian description of the bosonic sector of $N = 1$ supergravity [4] contains the vielbein $e_a{}^\mu$, the dilaton A, and a two form field $A_{a_1a_2}$. Introducing duals of both of the latter gauge fields, but not for the gravity, the complete field contents is given by

$$h_a{}^b\,, \qquad A\,, \qquad A_{a_1a_2}\,, \qquad A_{a_1\dots a_6}\,, \qquad A_{a_1\dots a_8}\,.$$

We want to identify these fields as the Goldstone bosons of a non-linear realisation. Thus the symmetry algebra must contain generators corresponding to the fields in (2), namely

$$K^a{}_b\,, \qquad R\,, \qquad R^{a_1a_2}\,, \qquad R^{a_1\dots a_6}\,, \qquad R^{a_1\dots a_8}\,. \tag{1.2}$$

We include the momentum generator P_a in order to introduce the notion of space-time into the algebra. We note that the $K^a{}_b$ generate the algebra A_9 which contains the Lorentz algebra with the generators $J_{ab} = K_{ab} - K_{ba}$, where we have raised and lowered indices with the flat Minkowski metric.

We take the generators introduced above to fulfill the following commutation relations

$$\begin{aligned}
[K^a{}_b, K^c{}_d] &= \delta^c_b K^a{}_d - \delta^a_d K^c{}_b\,, & [K^a{}_b, P_c] &= -\delta^a_c P_b\,,\\
[K^a{}_b, R^{a_1a_2}] &= \delta^{a_1}_b R^{aa_2} + \delta^{a_2}_b R^{a_1a}\,, & [K^a{}_b, R^{a_1\dots a_6}] &= \delta^{a_1}_b R^{aa_2\dots a_6} + \cdots\,,\\
[R, R^{a_1\dots a_p}] &= c_p R^{a_1\dots a_p}\,, & [R^{a_1a_2}, R^{b_1\dots b_6}] &= c_{2,6} R^{a_1a_2b_1\dots b_6}\,,
\end{aligned} \tag{2.1}$$

where $\dots$ stands for the appropriate anti-symmetrisation. We fix the three constants c to the values

$$c_2 = -c_6 = \frac{1}{2} = c_{2,6}\,. \tag{2.2}$$

All not-mentioned commutators (for example $[K^a{}_b, R]$) vanish. It is easy to check that these commutation relations satisfy the Jacobi identities. We call this algebra $G_{\rm Ips}$.

We take the Lorentz subalgebra to be a local symmetry while the whole algebra acts as a rigid symmetry. More specifically, if we take the general group element to be

$$g = \exp(x^\mu P_\mu)\, \exp(h_a{}^b K^a{}_b)\, g_A \equiv g_x\, g_h\, g_A, \tag{2.3}$$

where

$$g_A = \exp\left(\frac{1}{8!}A_{a_1\dots a_8}R^{a_1\dots a_8}\right)\, \exp\left(\frac{1}{6!}A_{a_1\dots a_6}R^{a_1\dots a_6}\right)\, \exp\left(\frac{1}{2}A_{a_1 a_2}R^{a_1 a_2}\right)\, \exp(AR)\,,$$

then we demand that the theory be invariant under

$$g \to g_0\, g\, h^{-1}\,,$$

where g_0 is an element of the whole group $G_{\rm Ips}$ and h an element of the local Lorentz transformations. This last equation means that we can always choose a particular Lorentz transformation to parametrise the resulting group element in terms of Goldstone fields alone. In turn the Maurer-Cartan form will always transform covariantly under the local Lorentz group.

We now calculate the Maurer-Cartan forms in the presence of the Lorentz connection $\omega = \frac{1}{2}dx^\mu \omega_{\mu\, b}{}^a J^b{}_a$:

$$\mathcal{V} = g^{-1}dg - \omega\,.$$

Since the Lorentz connection transforms as

$$\omega \to h\omega h^{-1} + hdh^{-1}\,,$$

we find that

$$\mathcal{V} \to h\mathcal{V}h^{-1}\,.$$

Defining

$$\mathcal{V} \equiv dx^\mu(e_\mu{}^a P_a + \Omega_{\mu\, a}{}^b K^a{}_b) + dx^\mu\left(\sum_{p=0,2,6,8}\frac{1}{p!}e^{-c_p A}\tilde{D}_\mu A_{a_1\dots a_p}R^{a_1\dots a_p}\right) \tag{2.4}$$

one finds that

$$e_\mu{}^a = (e^h)_\mu{}^a, \quad \Omega_{\mu\, a}{}^b = (e^{-1}\partial_\mu e)_a{}^b - \omega_{\mu\, a}{}^b\,.$$

The expression for $\tilde{D}_\mu A_{a_1\dots a_p}$ will be given below.

Pure $N=1$ supergravity is the non-linear realisation of the group that is the closure of the algebra $G_{\rm Ips}$ and the conformal algebra in ten dimensions. We therefore take only those combinations of the Cartan forms of $G_{\rm Ips}$ that can be rewritten as Cartan forms of the conformal group. This procedure was explained in reference [10] where it was shown that there was then a unique constraint on $\Omega_{ab}{}^c$ given by

$$\Omega_{a[bc]} - \Omega_{b(ac)} + \Omega_{c(ab)} = 0\,.$$

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In fact, it results in the usual expression of the spin connection in terms of the vielbeins of the theory. The $G_{\rm Ips}$ covariant derivative of the gauge field $A_{a_1a_2}$, for example, in equation (2.4) given by

$$\tilde{D}_\mu A_{a_1a_2} = \partial_\mu A_{a_1a_2} + (e^{-1}\partial_\mu)_{a_1}{}^b A_{ba_2} + (e^{-1}\partial_\mu)_{a_2}{}^b A_{a_1b}$$

is neither totally anti-symmetrised nor does it contain the derivatives that occur in general relativity. Demanding that the theory is invariant under the conformal group and $G_{\rm Ips}$ given in (2.1) and (2.2), we find that we must only use the totally anti-symmetrised Cartan forms of the gauge fields of equation (2.4). They are

$$\tilde{F}_a = \tilde{D}_a A = \partial_a A\,, \qquad \tilde{F}_{a_1a_2a_3} = 3e^{-1/2A}\tilde{D}_{[a_1}A_{a_2a_3]}$$

for the scalar and the 2-form potential respectively. Similarly we have

$$\tilde{F}_{a_1\dots a_7} = 7e^{1/2A}\tilde{D}_{[a_1}A_{a_2\dots a_7]}$$

for the 6-form potential and

$$\tilde{F}_{a_1\dots a_9} = 9(\tilde{D}_{[a_1}A_{a_2\dots a_9]} - 2\cdot 7A_{[a_1a_2}\tilde{D}_{a_3}A_{a_4\dots a_9]})$$

for the 8-form potential, where $\tilde{D}_a A_{a_1\dots a_p} = e_a{}^\mu(\partial_\mu A_{a_1\dots a_p} + (e^{-1}\partial_\mu e)_{a_1}{}^b A_{ba_2\dots a_p} + \dots)$ is the $G_{\rm Ips}$ covariant derivative, and $\dots$ indicates the action of $(e^{-1}\partial_\mu e)$ on all the other indices of the relevant gauge field. The equations of motion for the gauge fields can only be built from the covariant Cartan forms we have found and, assuming that they must result in equations that are of the usual order in space-time derivatives, they have to be

$$\tilde{F}^{a_1a_2a_3} = \frac{1}{7!}\epsilon^{a_1\dots a_{10}}\tilde{F}_{a_4\dots a_{10}}, \quad \text{and} \quad \tilde{F}^{a_1} = \frac{1}{9!}\epsilon^{a_1\dots a_{10}}\tilde{F}_{a_2\dots a_{10}}\,.$$

The only other non-trivial equation of motion involves the covariant derivatives of the spin-connection, namely the Riemann tensor

$$R_{\mu\nu a}{}^b \equiv \partial_\mu\omega_{\nu a}{}^b + \omega_{\mu a}{}^c\omega_{\nu c}{}^b - (\mu \leftrightarrow \nu)$$

or its contractions and is given by

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}\partial_{(\mu}A\partial_{\nu)}A + \frac{1}{4}g_{\mu\nu}(\partial A)^2 + \\ &+\frac{1}{16}e^{-A}\tilde{F}_{(\mu}{}^{\mu_1\mu_2}\tilde{F}_{\nu)\mu_1\mu_2} - \frac{1}{96}e^{-A}g_{\mu\nu}\tilde{F}^{\mu_1\mu_2\mu_3}\tilde{F}_{\mu_1\mu_2\mu_3} = 0\,. \end{aligned}$$

The combinatorial factors in front of each term in the last equation are not determined by the calculation above and we have simply inserted the correct values which are a consequence of supersymmetry. However, it is likely that the theory which is invariant under the full Kac-Moody algebra discussed later does have these coefficients fixed by the symmetry. The above are indeed the equations of motion of pure $N = 1$ supergravity [4].

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Our next task will be to embed the algebra G_{Ips} (where Ips stands for $N = 1$ pure supergravity) into a Kac-Moody algebra which, with suitable modifications to the theory, is expected to be a symmetry. The procedure was set out in reference [11] and as in this reference we exclude the translation P_a from this construction.. Let us therefore split the generators into

$$G^{+}_{\text{Ips}} = \{K^a{}_b,\, a < b,\, a, b = 1, \ldots, 10\,, \qquad R^{a_1 a_2},\;\; R^{a_1 \ldots a_6}\} \tag{2.5}$$

and the commuting generators

$$G^{0}_{\text{Ips}} = \left\{ H_a = K^a{}_a - K^{a+1}{}_{a+1},\, a = 1, \ldots, 9\,, \qquad D = \sum_a K^a{}_a,\;\; R \right\} \tag{2.6}$$

plus the remaining generators $K^a{}_b$, where $a > b$. The generators $K^a{}_b$ for $a < b$ are the positive roots of the SL(10) algebra, the generators H_a are its Cartan generators and the $K^a{}_b$, $a > b$ the negative roots. The Lorentz generators are the difference between positive and negative roots of SL(10). It was proposed in reference [11] that the local subgroup is that left invariant under the Cartan involution. This involution transforms positive roots (E_a) into minus the negative roots ($-F_a$) and negative roots (F_a) into minus the positive ones ($-E_a$), apart from $H_a \to -H_a$. The Lorentz generators are the subset of the SL(10)-generators which is invariant under the Cartan involution. In the bigger algebra G_{Ips} there will be other invariant generators thus contributing toward the local subgroup.

Kac-Moody algebras are defined in terms of their simple roots and their Cartan subalgebra. The positive (negative) roots are then given by taking repeated commutators of simple positive (negative) roots subject to the Serre relations. With the above choice of local subgroup, the positive roots of the Kac-Moody algebra must correspond to Goldstone fields in the non-linear realisation and as such the generators in (2.5) must be positive roots of the underlying Kac-Moody algebra. The set of generators in equation (2.5) can be found by taking multiple commutators of

$$E_a = K^a{}_{a+1}\,, \quad \text{for} \quad a = 1, \ldots, 9, \quad \text{and} \quad E_{10} = R^{9\,10}, \quad E_{-1} = R^{5\,6\,7\,8\,9\,10}\,. \tag{2.7}$$

We can identify these as simple roots of the Kac-Moody algebra. We also identify the generators of equation (2.6) as the Cartan subalgebra of the Kac-Moody algebra. We note that the number of simple roots is the same as the number of Cartan elements, namely eleven.

As explained in reference [11] to identify the Kac-Moody algebra requires further thought, as it does not contain most of the negative simple roots. To proceed further we consider certain subalgebras of G_{Ips}. One such subalgebra is of course A_9 corresponding to the $K^a{}_b$ generators of SL(10). Other simple subalgebras occur when restricting the range of the indices. One can easily check that the algebra (2.1) contains the Borel subalgebra of $D_6 \times A_1$, when the indices only assume values $i, j = 5, 6, 7, 8, 9, 10$. The D_6 Borel subalgebra is directly realised by the generators $K^i{}_j$ and R^{ij} and the 6-form generator which in six dimensions can be dualised to become a scalar and is then identified as the positive simple root of A_1. The seven Cartan generators needed to produce the Cartan matrix of $D_6 \times A_1$ are given uniquely and occur naturally. They will be given below.

As just seen, the elements of G_{Ips} contain the algebra A_9 and the (Borel subalgebra of) $D_6 \times A_1$. Since the $K^i{}_j$ generators of the latter coincide with part of the generators of the former (the A_5 subalgebra in D_6), it is simple to just extend the A_5 subalgebra to the full A_9. The additional four Dynkin nodes correspond to the indices on $K^i{}_j$ also assuming values $i, j = 1, 2, 3, 4$. It is less clear, how to connect the remaining A_1 node of the subalgebra $D_6 \times A_1$ to the resulting Dynkin diagram. We know, however, that in all known cases (including IIA and IIB supergravity in ten dimensions, eleven dimensional supergravity, and the 26 dimensional effective bosonic string action), supergravities dimensional restriction to two dimensions (indices assuming values $3, \ldots, d$) affinises the symmetry algebra. Assuming a similar mechanism to hold for the present case the transition from including seven dimensions to include eight dimensions then has to produce affine D_8. This fixes the additional A_1 node uniquely to be attached to the sixth node from the right. The whole symmetry algebra is then found to be very-extended D_8. A more complete argument for the occurrence of very-extended D_8 will be given shortly.

Before identifying other subgroups of G_{Ips}, the decomposition of $\mathrm{SO}(n, n)$ into representations of $\mathrm{SL}(n)$ is given. The reason for this will become apparent in due course. The $n(2n-1)$ generators of the former algebra decompose into generators $K^a{}_b$, D, R^{ab}, and R_{ab}. The corresponding degrees of freedom add up as

$$n(2n-1) = (n^2-1) + 1 + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} \tag{2.8}$$

respectively. The $K^a{}_b$ generators belong to the adjoint representation of $\mathrm{SL}(n)$ while D is the trace part of $GL(n)$. The two indexed generators belong to the two-index antisymmetric representation of $\mathrm{SL}(n)$. The generators R^{ab} and R_{ab} transform as 2-tensors under the $A_{n-1} \sim \mathrm{SL}(n)$ subalgebra. They can also be identified as part of the positive and negative roots of $\mathrm{SO}(n, n)$ respectively.

After this comment, we allow the indices of the algebra G_{Ips} in (2.1) to take values $i, j = 4, \ldots, 10$. We introduce the slightly redefined generators

$$\hat{K}^i{}_j = K^i{}_j - \frac{1}{8}\,\hat{D} + \frac{1}{2}\,R\,, \tag{2.9}$$

where $\hat{D} = \sum_{l=4}^{10} K^l{}_l$. We can dualise the six-form generator and introduce it into an extra generator by defining

$$\hat{K}^{-1}{}_i = S_i = \frac{1}{6!}\epsilon_{ii_1\ldots i_6} R^{i_1\ldots i_6}\,.$$

In order to enhance the Borel subalgebra of $\mathrm{SL}(7,\mathbb{R})$ to the Borel subalgebra of $\mathrm{SL}(8,\mathbb{R})$ we also need to define a new Cartan element:

$$\hat{K}^{-1}{}_{-1} = \frac{1}{8}\hat{D} - \frac{1}{2}R\,.$$

The index -1 corresponds to the new node in the Dynkin diagram required by the enhancement. Actually, we realise that the combination $\frac{1}{8}\hat{D} - \frac{1}{2}R$ is precisely the one that turns up in equation (2.9), so we can rewrite the definition of the hatted $\hat{K}$'s as

$$\hat{K}^i{}_j = K^i{}_j - \hat{K}^{-1}{}_{-1}.$$

These generators fulfill

$$[\hat{K}^i{}_j,\, \hat{K}^l{}_m] = \delta^l_j\, \hat{K}^i{}_m - \delta^i_m\, \hat{K}^l{}_j \quad \text{for} \quad i,j,l,m = -1,4,\ldots,10\,. \qquad (2.10)$$

The negative roots $K^i{}_{-1}$ are missing, but they are not part of the Borel sub-algebra we seek. We recognise(2.11) to be the commutation relations of SL$(8,\mathbb{R})$. However, we also have the anti-symmetric two-form generator R^{ij} fulfilling the relations

$$\begin{aligned}[][\hat{K}^i{}_j,\, R^{lm}] &= \delta^l_j\, R^{im} + \delta^m_j R^{li}\,,\\ [R^{ij},\, R^{lm}] &= 0\,.\end{aligned}$$

These relations together with (2.10) form the correct commutation relations of the Borel subalgebra of SO$(8,8)$ when written with respect to its SL$(8,\,\mathbb{R})$ subgroup provided we can achieve the index enhancement in the R^{lm} generators to also include the -1 index. In this case, we will have shown that the Borel subalgebra of SO$(8,8)$ is a symmetry of the theory. At first glance the supergravity theory does not seem to have any further fields though. It has been discussed in references [11, 12] that for the theory to be invariant under a Kac-Moody algebra requires it to be formulated in a way that treats the gravity and gauge field sectors of the theory on an equal footing. This requires a formulation of gravity that mirrors the dual formulation employed for the gauge fields. As explained in reference [11] one must introduce in D dimensions the field $A_{a_1\ldots a_{D-3}}{}^b$ along with the usual field $h_{a_1}{}^b$. Thus in ten dimensions we would have to introduce $A_{a_4\ldots a_{10}}{}^b$. More evidence for the occurrence of this field was given in references [11, 12] for the cases of eleven dimensional supergravity, and the two maximal (non-massive) supergravities in ten dimensions: IIA and IIB. In reference [11] it was shown that there is indeed a dual formulation of gravity which on the linearised level involves the fields $h_{a_1}{}^b$ and $A_{a_4\ldots a_{10}}{}^b$.

As such, we introduce the corresponding generator $R^{a_4\cdots a_{10},b}$ which occurs in the commutation relation between the six and the two-form potential (2.1) which we modify to now be

$$\left[R^{a_1\ldots a_6},\, R^{b_1b_2}\right] = -c_{2,6}R^{a_1\ldots a_6b_1b_2} + \frac{2}{7}\, R^{a_1\ldots a_6[b_1,b_2]}\,.$$

The coefficient in front of the last generator in this equation can be chosen by appropriately rescaling this generator and we choose $\frac{2}{7}$ for later convenience. The change of this commutation relation is similar to the cases of IIA and IIB supergravity in ten dimensions.

We now investigate the consequences of this new generator for the restriction discussed above. In particular, we can dualise this new generator $T^j = \frac{1}{7!}\epsilon_{i_4\ldots i_{10}}R^{i_4\cdots i_{10},j}$ and identify

$$R^{-1j} = T^j = \frac{1}{7!}\epsilon_{i_4\ldots i_{10}}R^{i_4\ldots i_{10},j}\,.$$

The commutators of this new generator with the generators $K^i{}_j$, $K^{-1}{}_j$, and R^{-1i} follow trivially and indeed lead to the correct commutation relations for the Borel subalgebra of SO$(8,8)$; namely

$$[K^{-1}{}_i,\, R^{jk}] = \delta^j_i\, R^{-1k} - \delta^k_i\, R^{-1j}\,.$$

The unique Cartan generators which lead to the Dynkin diagram of D_8 are given by

$$\begin{aligned} H_a &= K^a{}_a - K^{a+1}{}_{a+1} \quad \text{for } a = 4, \ldots, 9\,, \\ H_{10} &= K^9{}_9 + K^{10}{}_{10} - 2\hat{K}^{-1}{}_{-1}, \quad H_{-1} = -K^4{}_4 + 2\hat{K}^{-1}{}_{-1}\,. \end{aligned} \tag{2.11}$$

We now show that the required Kac-Moody algebra that contains $G_{\rm Ips}$ is very-extended D_8 by finding the Kac-Moody algebra that contains two subalgebras A_9 and D_8 in the required way. Since we know how six of the nodes of the A_9 subalgebra are contained in the D_8 subalgebra we can attach the three additional nodes in A_9 (indices also take values 1, 2, 3) to the D_8 Dynkin diagram in a line so that they also form the A_9 algebra in the correct way. We are then left to find the number of connections of these new nodes to the two nodes that represent the 2-form and the 6-form potential.

To resolve this point we need to study the effect of adding a node to the Dynkin diagram of A_{r-1} to define the Dynkin diagram of a new algebra g. Deleting this node obviously leaves us with the algebra A_{r-1} and so we are in the situation considered in section three of reference [15]. We may write the root α_c corresponding to the additional node as $\alpha_c = \sum_i A_{ci}\frac{(\alpha_c,\alpha_c)}{2}\lambda_i + x$ in the notation of reference [15] which only considered the case of simply laced algebras. The adjoint representation of the enlarged algebra g clearly contains the root α_c and so the A_{r-1} representation with Dynkin indices $p_j = -\frac{(\alpha_c,\alpha_c)}{2}A_{cj}$. Indeed, the simple root E_c corresponding to the additional node is part of this representation. In particular, if the additional node attaches with one line to the pth node from the right in the Dynkin diagram of A_{r-1}, the generator corresponding to the simple root belongs to the rank p anti-symmetric tensor of representation A_{r-1}. If we consider the notion of level introduced in reference [15], i.e. the number of times α_c occurs in a root of g, then clearly α_c is at level one. However, at higher levels one will in general find additional representations of A_{r-1} in the adjoint representation of the enlarged algebra g. These can be determined at low levels by considering the constraints discussed in reference [15].

Hence the way a node attaches to the A_{r-1} line is fixed by the A_{r-1} representation that the corresponding simple root of the node belongs to. One example is in eleven dimensional supergravity where the generator corresponding to the 3-form potential attaches to the third node from the right of the A_{10} line and thus provides the typical shape of an exceptional Lie algebra.

Returning to the present case of heterotic supergravity in ten dimensions, we see that the simple positive root $E_{10} = R^{9\,10}$ of $G_{\rm Ips}$ in (2.7) belongs to the rank 2 anti-symmetric tensor representation of the A_9 algebra. As a result, this node must attach only to the second node from the right of the A_9 line. As such, it cannot attach to the three nodes of A_9 which are not in D_8.

A very similar argument holds for the generator corresponding to the 6-form potential $R^{5678910}$ in equation (2.7) which in the heterotic case is a simple root since it cannot be built up by taking commutators of other simple roots. It belongs to the sixth rank anti-symmetric representation of A_9 and as such can only attach to the sixth node from the right of the A_9 line.

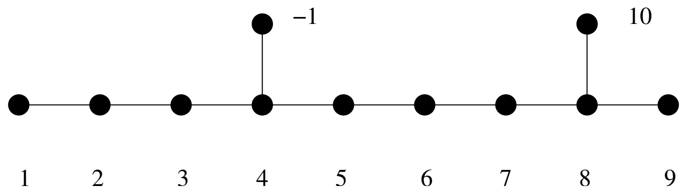

Figure 1: The Dynkin diagram of very-extended D_8.

As a result we may conclude that the Kac-Moody algebra we are searching is uniquely given by very-extended D_8 depicted in figure 1. Of course this algebra contains an infinite number of generators and so we expect an infinite number of fields in the corresponding non-linear realisation. It is hoped that these will lead to new propagating degrees of freedom.

We find the A_9 subgroup as a horizontal line, and the 2-form potential (the node denoted 10) sticking out from the eighth node, and the 6-form potential (denoted -1) from the fourth node.

The Cartan generators leading to the usual form of the Cartan matrix of very-extended D_8 are given by

$$\begin{aligned} H_a &= K^a{}_a - K^{a+1}{}_{a+1} \quad \text{for } a = 1, \ldots, 9\,, \\ H_{10} &= K^9{}_9 + K^{10}{}_{10} - 2K^{-1}{}_{-1}\,, \quad H_{-1} = -K^1{}_1 - K^2{}_2 - K^3{}_3 - K^4{}_4 + 2K^{-1}{}_{-1} \end{aligned} \tag{2.12}$$

The $K^{-1}{}_{-1}$ generator is left unhatted since it contains the full trace $D = \sum_{a=1}^{10} K^a{}_a$ (compare equation (2.9). By construction, the Dynkin diagram of the D_8 subalgebra appears very naturally in figure 1 when considering the indices $-1, 4, \ldots, 10$ only.

Since $G_{\rm Ips}$ is a symmetry of ten dimensional supergravity it follows that the Borel subalgebra of SO(8, 8) is also a symmetry once one had succeeded in incorporating the duals of gravity. The positive roots R^{ab} of the decomposition (2.8) correspond to Goldstone bosons of the non-linear realisation. The negative root counterparts R_{ab} do not have any field analogues and therefore contribute towards the local symmetry group, and thus should be hidden symmetries of the equations of motion.

It is instructive to check consistency with the similar calculations in IIA supergravity theory in 10 dimensions. The field content of IIA supergravity including the gauge field duals is given by

$$h_a{}^b,\ A_a,\ A_{a_1a_2},\ A_{a_1a_2a_3},\ A_{a_1\ldots a_5},\ A_{a_1\ldots a_6},\ A_{a_1\ldots a_7},\ A_{a_1\ldots a_8}\,.$$

In pure $N = 1$ supergravity, we do not have gauge vectors or three forms, neither do we have their duals: the seven- or the five form. It is natural to define an involution $\mathcal{I}$ on the corresponding generators which acts as

$$\mathcal{I}: K^a{}_b \to K^a{}_b\,, \quad \mathcal{I}: R \to R, \mathcal{I}: R^{a_1a_2} \to R^{a_1a_2}\,, \quad \mathcal{I}: R^{a_1\ldots a_6}\,, \quad \mathcal{I}: R^{a_1\ldots a_8} \to R^{a_1\ldots a_8}\,,$$

but also

$$\mathcal{I}: R^a \to -R^a, \mathcal{I}: R^{a_1a_2a_3} \to -R^{a_1a_2a_3}, \mathcal{I}: R^{a_1\ldots a_5} \to -R^{a_1\ldots a_5}, \mathcal{I}: R^{a_1\ldots a_7} \to -R^{a_1\ldots a_7}\,.$$

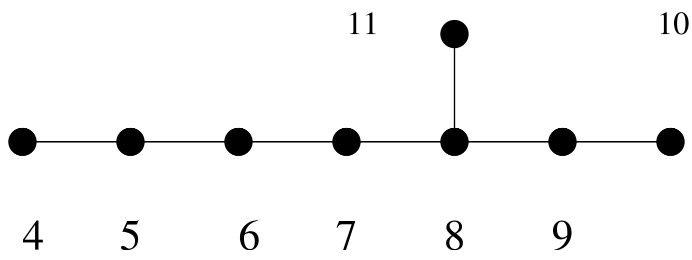

Figure 2: The Dynkin diagram of E_8.

Clearly, $\mathcal{I}^2 = I$ and the invariant fields are those of $N = 1$ supergravity. We note that the six-form generator which is fundamental to $N = 1$ supergravity in ten dimensions since it is a simple positive root, also turns up in IIA supergravity. However, in the latter theory it is not fundamental but can be build up by taking repeated commutation relations between the simple positive roots of IIA which are given by

$$K^a{}_{a+1} \quad \text{for } a = 1, \ldots, 9, \quad R^{10}, \quad \text{and } R^{9\,10}\,.$$

Using the G_{IIA} commutation relations

$$[R^{a_1\ldots a_p},\, R^{a_{p+1}\ldots a_{p+q}}] = c_{p,q} R^{a_1\ldots a_{p+q}}$$

with non-trivial coefficients $c_{1,2}$ and $c_{3,3}$ (amongst others) and the tensor transformation behaviour under SL(10) encoded in

$$[K^a{}_b,\, R^{a_1\ldots a_p}] = p\, \delta_b^{[a_1} R^{|a|a_2\ldots a_p]}\,,$$

we find that we can build up the 6-form generator $R^{56789\,10}$ by taking repeated commutation relations of the IIA simple positive roots

$$2\times R^{10},\, 2\times R^{9\,10},\, 3\times K^9{}_{10},\, 4\times K^8{}_9,\, 3\times K^7{}_8,\, 2\times K^6{}_7, \text{ and } \; K^5{}_6\,.$$

We do not consider the simple positive root R^{10} in the IIA algebra since it is not invariant under the involution $\mathcal{I}$, but we introduce a new simple positive root corresponding to the 6-form potential which is invariant under $\mathcal{I}$. This new simple positive root $\alpha_\#$ can thus be written as

$$\alpha_\# = 2\alpha_{10} + 2\alpha_{11} + 3\alpha_9 + 4\alpha_8 + 3\alpha_7 + 2\alpha_6 + \alpha_5\,,$$

where the subscripts correspond to the labels in the Dynkin diagram of E_8 (figure 2). The E_{11} of the IIA algebra is just very extended E_8 and the embedding of the root into the larger algebra is trivial.

We find that the new root satisfies $\alpha_\#^2 = 2$, while all the relations with other roots (but cancelling α_{10}, which in IIA is introduced via $K^a{}_{11} = 2R^a$, and refers to the vector generator which is not invariant under $\mathcal{I}$) give the root system of D_8.

We conclude this chapter by noting that the same calculation goes through for any dimension d of the very-extended D_{d-2} Kac-Moody algebra. The corresponding gravity theory comprises gravity, as well as a dilaton and a two-form gauge field along with their duals. One example is the case of an effective description of purely bosonic string theory in 26 dimensions, which has the same field content, to give a more streamlined derivation of the Kac-Moody algebra K_{27}, or very-extended D_{24} [16].

3. $N = 1$ supergravity with one vector multiplet

In this section we extend the discussion of the previous section to the case of $N = 1$ supergravity coupled to one abelian $N = 1$ super vector multiplet. The bosonic field content of this theory is that of the pure supergravity case, but we have one additional vector field A_a. We denote the corresponding group in the non-linear realisation by G_{Ia}, where the Ia reminds us that we are still dealing with an $N = 1$ theory with an abelian extension. We introduce the Hodge dual of this vector field, which in ten dimensions can only be a seven-form field. Introducing generators to all gauge fields, we can write the general group element of G_{Ia} as

$$g = \exp(x^\mu\, P_\mu)\, \exp(h_a{}^b\, K^a{}_b)\, g_A \equiv g_x\, g_h\, g_A\,,$$

where g_A is now slightly modified in comparison to (2.3) by the additional term containing the abelian gauge field

$$\begin{aligned} g_A = \exp\left(\frac{1}{8!}A_{a_1\ldots a_8}R^{a_1\ldots a_8}\right)\, \exp\left(\frac{1}{7!}A_{a_1\ldots a_7}R^{a_1\ldots a_7}\right)\, \exp\left(\frac{1}{6!}A_{a_1\ldots a_6}R^{a_1\ldots a_6}\right)\times \\ \times \exp\left(\frac{1}{2}A_{a_1a_2}R^{a_1a_2}\right)\, \exp\left(A_aR^a\right)\, \exp(AR)\,. \end{aligned}$$

This group element contains all the fields of the theory, but it also contains the Lorentz group as the antisymmetric part of the K_{ab} generators. As usually, we have added in the momentum generator.

The generators are taken to fulfill the following commutation relations

$$\begin{aligned} [K^a{}_b,\, K^c{}_d] &= \delta^c_b K^a{}_d - \delta^a_d K^c{}_b\,, & [K^a{}_b,\, P_c] &= -\delta^a_c P_b\,, \\ [K^a{}_b,\, R^c] &= \delta^c_b R^a\,, & [K^a{}_b,\, R^{a_1\ldots a_p}] &= \delta^{a_1}_b R^{aa_2\ldots a_p} + \cdots\,, \\ [R,\, R^{a_1\ldots a_p}] &= c_p R^{a_1\ldots a_p}\,, & [R^{a_1\ldots a_p},\, R^{b_1\ldots b_q}] &= c_{p,q} R^{a_1\ldots a_p b_1\ldots b_q}\,, \end{aligned} \tag{3.1}$$

where $\cdots$ in the second line stands for the appropriate anti-symmetrisations. Since the case of pure supergravity is a truncation of the present one, the constants c_p and $c_{p,q}$ should contain those of pure supergravity (equation (2.2)). We adopt those old values:

$$c_1 = -c_7 = \frac{1}{4}\,, \qquad \text{and} \quad c_2 = -c_6 = \frac{1}{2}\,,$$

but also:

$$c_{1,1} = -2\,, \qquad c_{1,6} = 1\,, \qquad c_{1,7} = -\frac{1}{2}\,, \qquad \text{and} \qquad c_{2,6} = \frac{1}{2}\,. \tag{3.2}$$

Using these commutation relations we can presently calculate the Maurer-Cartan forms in the presence of the Lorentz connection (similarly to the previous chapter):

$$\mathcal{V} \equiv dx^\mu (e_\mu{}^a P_a + \Omega_{\mu\, a}{}^b K^a{}_b) + dx^\mu \left(\sum_{p=0,1,2,6,7,8} \frac{1}{p!} e^{-c_p A} \tilde{D}_\mu A_{a_1\ldots a_p} R^{a_1\ldots a_p}\right).$$

After taking the closure with the conformal group, the objects in front of the gauge generators are totally anti-symmetrised. They are given by

$$\begin{aligned}
\tilde{F}_a &= \tilde{D}_a A, \\
\tilde{F}_{a_1 a_2} &= 2e^{-1/4A} \tilde{D}_{[a_1} A_{a_2]}, \\
\tilde{F}_{a_1 a_2 a_3} &= 2e^{-1/2A} \left(\tilde{D}_{[a_1} A_{a_2 a_3]} - 2A_{[a_1} \tilde{D}_{a_2} A_{a_3]} \right)
\end{aligned}$$

for the original fields, and for the duals of those fields by

$$\begin{aligned}
\tilde{F}_{a_1 \dots a_7} &= 7e^{1/2A} \tilde{D}_{[a_1} A_{a_2 \dots a_7]}, \\
\tilde{F}_{a_1 \dots a_8} &= 8e^{1/4A} \left(\tilde{D}_{[a_1} A_{a_2 \dots a_8]} - 7A_{[a_1} \tilde{D}_{a_2} A_{a_3 \dots a_8]} \right)
\end{aligned}$$

and

$$\tilde{F}_{a_1 \dots a_9} = 9 \left(\tilde{D}_{[a_1} A_{a_2 \dots a_9]} - 2 \cdot 7A_{[a_1 a_2} \tilde{D}_{a_3} A_{a_4 \dots a_9]} + 4A_{[a_1} \tilde{D}_{a_2} A_{a_3 \dots a_9]} \right).$$

These field strengths are the correct field strengths of $N = 1$ supergravity with one abelian vector added in. The equations of motion can be found by demanding that these field strengths have to be related to each other in a Lorentz covariant manner. As such, they can only be

$$\tilde{F}^{a_1 \dots a_p} = \frac{1}{(10-p)!} \epsilon^{a_1 \dots a_{10}} \tilde{F}_{a_{p+1} \dots a_{10}}, \quad \text{for } p = 1, 2, 3. \tag{3.3}$$

These are the correct field equations of the $N = 1$ supergravity theory coupled to an abelian vector multiplet. As in the preceding section it can be shown that also the Einstein equation is given uniquely (except for some coefficients) after taking the closure with the conformal group and adopting the inverse Higgs constraint $\Omega_{a[bc]} - \Omega_{b(ac)} + \Omega_{c(ab)} = 0$. The Einstein equation with fitted coefficients then reads

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} \partial_{(\mu} A \partial_{\nu)} A + \frac{1}{4} g_{\mu\nu} (\partial A)^2 - 2 \left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4} G_{\mu\nu} F^2 \right) + \\
+ \frac{1}{16} e^{-A} \tilde{F}_{(\mu}{}^{\mu_1 \mu_2} \tilde{F}_{\nu) \mu_1 \mu_2} - \frac{1}{96} e^{-A} g_{\mu\nu} \tilde{F}^{\mu_1 \mu_2 \mu_3} \tilde{F}_{\mu_1 \mu_2 \mu_3} = 0.
\end{aligned}$$

We conclude that the above group G_{Ia} indeed is a symmetry of the theory.

Assuming the theory is invariant under a Kac-Moody algebra we now wish to identify it. In order to do so, and after excluding the translations P_a from this discussion, we divide the generators into

$$G_{Ia}^+ = \{ K^a{}_b,\, a < b,\, a, b = 1, \dots, 10, \quad R^a,\, R^{a_1 a_2},\, R^{a_1 \dots a_6},\, R^{a_1 \dots a_7},\, R^{a_1 \dots a_8} \}. \tag{3.4}$$

We identify these as some of the positive roots of the underlying Kac-Moody algebra. Similarly, the generators of its Cartan subalgebra are taken to be

$$G_{Ia}^0 = \left\{ K^a{}_a,\, a = 1, \dots, 10,\, D = \sum_a K^a{}_a, \quad R \right\}.$$

The only generators that are missing form equation (3.1) are the negative roots of the $K^a{}_b$ algebra, where $a > b$.

Again we can identify a unique set of simple positive roots by finding those whose repeated commutators give all those in G_{Ia}^{+}. These simple roots are

$$E_a = K^a{}_{a+1} \quad \text{for} \quad a = 1, \ldots, 9, \quad E_{10} = R^{10}, \quad E_{-1} = R^{56789\,10}.$$

Apart from the obvious SL(10, $\mathbb{R}$) subgroup we can also find a $B_6 \times A_1$ subgroup when we restrict the indices to only assume six values $i, j = 5, \ldots, 10$. The seven and eight form generators get projected out in this restriction. Indeed, the Borel subalgebra of $B_6 \times A_1$ is realised in the commutation relations of the field generators if we slightly rescale the vector generator $R^i \to \frac{R^i}{\sqrt{2}}$. We then find

$$\begin{aligned}
&[K^i{}_j, K^l{}_m] = \delta^j_l\, K^i{}_m - \delta^i_m\, K^l{}_j \\
&[K^i{}_j, R^l] = \delta^l_j\, R^i\,, \qquad [K^i{}_j, R^{lm}] = \delta^l_j\, R^{im} + \delta^m_j R^{li}\,, \\
&[R^i, R^{lm}] = 0\,, \qquad [R^i, R^j] = -R^{ij} \qquad (\textit{here:}\ \text{all indices } 5, \ldots, 10)\,.
\end{aligned}$$

As in the case of pure supergravity, the six form generator $R^{56789\,10}$ is the (simple) positive root of the A_1 which decouples from the B_6 algebra part.

In what follows we will need the decomposition of SO$(n, n+1)$ into representations of SL(n). The $n(2n+1)$ generators of the former split up into generators $K^a{}_b$, D, R^{ab} and R_{ab}, but also R^a, and R_a (indices running from $1, \ldots, n$). The corresponding degrees of freedom add up as

$$n(2n+1) = (n^2 - 1) + 1 + 2 \times \frac{n(n-1)}{2} + 2 \times n\,,$$

respectively. The R^{ab} and R_{ab} generators again form the rank-2 antisymmetric representation, but now we also have rank-1 contributions from R^a and R_a. These objects transform as vectors and co-vectors under SL(n). The R-generators with upper and lower indices belong to the positive and negative roots of B_n respectively. Given our choice of local subalgebra the positive root generators lead to Goldstone bosons in the non-linear realisation.

We show next, that we can in fact find the Borel subalgebra of B_8 which has the commutation relations

$$\begin{aligned}
&[K^a{}_b, K^c{}_d] = \delta^c_b\, K^a{}_d - \delta^a_d\, K^c{}_d \\
&[K^a{}_b, R^c] = \delta^c_b\, R^a\,, \qquad [K^a{}_b, R^{cd}] = \delta^c_b\, R^{ad} + \delta^d_b R^{ca}\,, \\
&[R^a, R^{bc}] = 0\,, \qquad [R^a, R^b] = -R^{ab} \qquad (\textit{here:}\ \text{all indices } 1, \ldots, 8)
\end{aligned} \tag{3.5}$$

from the commutation relations of G_{Ia} after a suitable rescaling. We start by allowing seven values for the indices $i, j = 4, \ldots, 10$. The new simple positive root is called $E_4 = K^4{}_5$, the new Cartan element has to be $H_4 = K^4{}_4 - K^5{}_5$ in order to find the correct SL(7, $\mathbb{R}$) subgroup (the commutators with the positive root of A_1 and E_{10} ensure that there are no trace or dilaton parts $\hat{D}$ or R). We note that $[H_4, E_4] = 2\,E_4$ as it must be, and $[H_4, E_5] = -E_5$. We also find $[H_4, E_{-1}] = -E_{-1}$, which tells us that the new node couples the extra node A_1 to the existing B_6 diagram to give B_8. We define the generators

$$\hat{K}^{-1}{}_i = \frac{1}{6!}\epsilon_{ii_1\ldots i_6} R^{i_1\ldots i_6}$$

and by dualising the seven-form generator we extend the vector generator:

$$\hat{R}^{-1} = -\frac{7}{\sqrt{2}\,7!}\epsilon_{i_1\ldots i_7}R^{i_1\ldots i_7}\,.$$

We seem to have run out of Goldstone fields in the theory to also enhance the R^{ab}-generator. As realised previously, we must introduce duals of gravity to find the missing Goldstone fields (see chapter 1). This happens by altering the existing commutation relations (3.2) to now become

$$\left[R^{c_1\ldots c_6},\, R^{bc}\right] = -c_{2,6}R^{c_1\ldots c_6 bc} + \frac{2}{7}R^{c_1\ldots c_6[b,c]}\,.$$

The Jacobi identities then imply the relation

$$\left[R^{c_1\ldots c_7},\, R^{b}\right] = -c_{1,7}R^{c_1\ldots c_7 b} + \frac{2}{7}R^{c_1\ldots c_7,b}\,.$$

This new generator, corresponding to dual of gravity, can in seven dimensions be dualised and used to define

$$\hat{R}^{-1\,j} = \frac{1}{7!}\epsilon_{i_1\ldots i_7}R^{i_1\ldots i_7,j}\,.$$

We also have to introduce a new generator for the Cartan subalgebra

$$\hat{K}^{-1}{}_{-1} = a\,\hat{D} + b\,R, \quad \hat{D} = \sum_{l=4}^{10} K^l{}_l\,.$$

The constants can be fixed from the commutation relations with other generators. In particular we require

$$\left[\hat{K}^{-1}{}_{-1},\, R^{ij}\right] = 0\,, \qquad \left[\hat{K}^{-1}{}_{-1},\, \hat{K}^{-1}{}_{j}\right] = \hat{K}^{-1}{}_{j}\,, \qquad \left[\hat{K}^{-1}{}_{-1},\, \hat{R}^{-1\,i}\right] = \hat{R}^{-1\,i}$$

and thus find that $a = 1/8$ and $b = -1/2$. We have no degrees of freedom left, but it is a non-trivial check that for these values we yield the right commutation relations for the vector

$$\left[\hat{K}^{-1}{}_{-1},\, \hat{R}^{i}\right] = 0\,, \qquad \left[\hat{K}^{-1}{}_{-1},\, \hat{R}^{-1}\right] = \hat{R}^{-1}\,,$$

as indeed we do. Using the slightly modified generators

$$\hat{K}^i{}_j = K^i{}_j - \frac{1}{8}\,\hat{D} + \frac{1}{2}\,R = K^a{}_b - \hat{K}^{-1}{}_{-1}\,,$$

one can check that these generators indeed fulfill the correct commutation relations of the Borel subalgebra of B_8 when written in terms of representations of $\mathrm{SL}(8,\mathbb{R})$ as given in equation (3.5) if the eight indices are taken to be $i, j = -1, 4, \ldots, 10$.

Since G_{Ia} is a symmetry of the supergravity theory with one abelian vector it follows that also the Borel subalgebra of $\mathrm{SO}(8,9)$ is a symmetry once we have found a formulation involving the duals of gravity. A theory invariant under the full B_8 will have to include the symmetries under the negative roots R_{ab} and R_a. These symmetries are understood to be unbroken local symmetries and therefore do not contribute towards the Goldstone spectrum. They enhance the local Lorentz symmetry ($\mathrm{SO}(8)$) and should correspond to hidden symmetries of the equations of motion.

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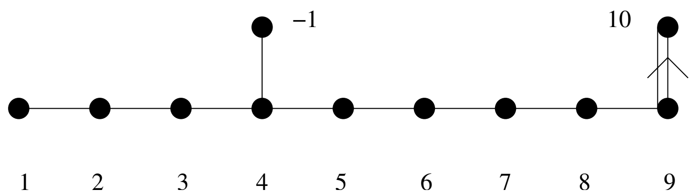

Figure 3: The Dynkin diagram of very-extended B_8.

We know that the ten dimensional theory must have $\mathrm{SL}(10, \mathbb{R})$ as a symmetry group too. We summarise that the Kac-Moody group we want to identify needs to have B_8 and A_9 as subalgebras when accordingly restricted. The algebra of B_8 occurred when the indices assumed the seven values $i, j = 4, \ldots, 10$. It naturally included (see (3.5)) the $K^i{}_j$ generators of the A_6 subalgebra. We now have to enhance this A_6 subalgebra to the full A_9 subalgebra and clarify how the additional three nodes couple to the rest of the Dynkin diagram. Looking at the positive roots (3.4) of the G_{Ia} algebra, and in particular at the simple positive roots (3), we find that the generators corresponding to the vector potential and 6-form potential transform as rank one and six anti-symmetric tensors under the A_9 subalgebra (3.1). The commutations of equations (3.1) imply that the 6-form generator $R^{a_1 \ldots a_6}$ and the vector generator R^a transform as their indices indicate under A_9. Using the discussion from the chapter 1 concerning the addition of extra nodes to the Dynkin diagram of A_n, we conclude that the corresponding simple generators they contain can only attach respectively to the first and sixth nodes from the right of the A_9 line. In particular, there are no additional connections between the nodes 1, 2, 3 and the nodes -1, 10. This resulting group is very-extended B_8, whose Dynkin diagram is given in figure 3. We also note that very-extended B_8 contains affine B_8 if the indices do not assume values 1, 2.

The Chevalley generators of the Kac-Moody algebra have as their simple positive roots

$$E_a = K^a{}_{a+1} \quad \text{for} \quad a = 1, \ldots, 9, \quad E_{10} = R^{10}, \quad E_{-1} = R^{56789\,10}$$

as well as the Cartan sub-algebra elements

$$\begin{aligned} H_a &= K^a{}_a - K^{a+1}{}_{a+1} \quad \text{for} \quad a = 1, \ldots, 9\,, \qquad H_{10} = 2K^{10}{}_{10} - \frac{1}{4} D + \frac{1}{2} R\,, \\ H_{-1} &= -K^1{}_1 - K^2{}_2 - K^3{}_3 - K^4{}_4 + 2K^{-1}{}_{-1}\,, \end{aligned} \tag{3.6}$$

where the unhatted $D = \sum_{l=1}^{10} K^l{}_l$ runs over all ten values (also in the definition of $K^{-1}{}_{-1} = \frac{1}{8} D - \frac{1}{2} R$). Except for H_{10}, and E_{10} these generators coincide with those of pure supergravity described in the previous chapter.

4. $N = 1$ supergravity with more than one vector multiplet

The classification of semi-simple Lie algebras is usually carried out when the algebras are taken to be over the complex numbers. Consequently, the end result of the classification, i.e. the Dynkin diagram, does not specify a preferred real form of the algebra. Any finite dimensional complex Lie algebra possesses a unique real form in which all the generators are compact. It is given by taking the generators $U_\alpha = i(E_\alpha + E_{-\alpha})$, $V_\alpha = (E_\alpha - E_{-\alpha})$ and iH_a, where α is any positive root. The compact nature of the generators follows in an obvious way form the fact that $(E_\alpha, E_{-\alpha}) > 0$. By considering all involutive automorphisms of the unique compact real form one can construct all other real forms of the complex Lie algebra under consideration. In particular, the real forms are in one to one correspondence with all those involutive automorphisms of the compact real algebra which can not be written as real transformations.

Given an automorphism $\mathcal{I}$ which is involutive ($\mathcal{I}^2 = 1$) we can divide the generators of the compact real form into those which possess $+1$ and -1 eigenvalues of $\mathcal{I}$. We denote these eigenspaces by

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$$

respectively. Since $\mathcal{I}$ is an automorphism, the algebra when written in terms of this split takes the generic form

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{P}] = \mathcal{P}\,, \qquad [\mathcal{P}, \mathcal{P}] \subset \mathcal{K}.$$

For the generators $\mathcal{P}$ we define new generators $\hat{\mathcal{P}} = i\mathcal{P}$, hence the algebra takes the generic form

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}\,, \qquad [\mathcal{K}, \hat{\mathcal{P}}] = \hat{\mathcal{P}}\,, \qquad [\hat{\mathcal{P}}, \hat{\mathcal{P}}] \subset (-1)\mathcal{K}\,.$$

Thus we find a new real form of the algebra in which the generators $\mathcal{K}$ are compact while the generators $\hat{\mathcal{P}}$ are non-compact. This follows from the fact that all the generators in the original algebra are compact and so have negative definite scalar product. Hence, the number of non-compact generators changes from one real form to another.

An important real form can be constructed by considering the Cartan involution which is a linear operator that takes $E_\alpha \leftrightarrow -E_{-\alpha}$ and $H_a \rightarrow -H_a$. Clearly the generators of the compact real form transform as $V_\alpha \rightarrow V_\alpha$, $U_\alpha \rightarrow -U_\alpha$ and $H_a \rightarrow -H_a$. Using this involution we find a real form with generators

$$\hat{V}_\alpha = V_\alpha = E_\alpha - E_{-\alpha}\,, \qquad \hat{U}_\alpha = -iU_\alpha = E_\alpha + E_{-\alpha}\,, \qquad \hat{H}_a = -iH_a\,.$$

The $\hat{V}_\alpha$ remain compact generators while $\hat{U}_\alpha$ and $\hat{H}_a$ become non-compact. The maximal compact sub-algebra is just that invariant under the Cartan involution. The real form of the algebra found in this way has the maximal number of non-compact generators of all the real forms one can construct. It is therefore called the maximal non-compact real form. For example, the complex Lie algebra D_n has $\mathrm{SO}(2n)$ as its unique compact real form and $\mathrm{SO}(n, n)$ as its maximal non-compact real form. For a more detailed discussion of real forms see [17].

Given a real form of a complex Lie algebra, any element g of the associated group can be expressed as $g = g_c g_{na} g_r$ where g_c is in the maximal compact sub-group, g_{na} is the maximal commuting non-compact subalgebra and g_r is the group found by exponentiating the generators which are in the positive root space of g_{na}. For the case of a maximally non-compact form of the algebra, all the Cartan generators are non-compact and so g_{na} is just the Cartan sub-algebra while g_r is generated by all positive root generators of the original algebra. As such this is the decomposition for which $g_{na}g_r$ is just the Borel sub-group. The important point to note is that it is only for the maximally non-compact real form that all Cartan subalgebra elements of the original algebra appear in g_{na} and all simple roots can be generated from multiple commutators of the generators appearing in g_r.

The construction of non-linear realisations based on a given algebra is carried out with respect to a particular real form of a given algebra. An important ingredient in the construction is the choice of local subalgebra as this effects the field content and the way the symmetries are realised. The local subalgebra is usually chosen to be the maximally compact subalgebra of a the real form being used. Clearly, this subalgebra changes from one real form to another. For example, for $\mathrm{SO}(n,n)$ the maximally compact sub-algebra is $\mathrm{SO}(n)\otimes\mathrm{SO}(n)$ while for $\mathrm{SO}(p,q)$ it is $\mathrm{SO}(p)\otimes\mathrm{SO}(q)$. Clearly , even if $p+q=2n, p\neq q$ the dimensions of the two cosets are different and so is the physics resulting from the two non-linear realisations based on the two algebras. In particular, only for $\mathrm{SO}(n,n)$ do all the Cartan subalgebra generators appear in the coset. So far, all algebras considered in the context of the eleven dimensional supergravity, IIA and IIB supergravity and the $N=1$ supergravity theory coupled to no or one vector multiplet were the maximal non-compact form of real algebras. However, we will see that with more than one vector multiplet coupled to $N=1$ supergravity one must consider symmetry algebras that are not the maximally non-compact real form.

The field contents of $N=1$ supergravity with an arbitrary number of abelian vectors n added is given by

$$h_a{}^b\,,\qquad A,\quad A^{(k)a}\,,\qquad A^{a_1a_2}\,,\qquad A^{a_1\dots a_6}\,,\qquad A^{(k)a_1\dots a_7}\,,\qquad A^{a_1\dots a_8}\,,$$

where the indices a_i, b take ten values according to the ten dimensional tangent space, while the index (k) numerates the abelian vectors $1,\dots,n$. Since we wish to express the theory as a non-linear realisation we introduce the corresponding generators

$$K^a{}_b\,,\qquad R\,,\qquad R^{(k)}_a\,,\qquad R_{a_1a_2}\,,\qquad R_{a_1\dots a_6}\,,\qquad R^{(k)}_{a_1\dots a_7}\,,\qquad R_{a_1\dots a_8}\,. \tag{4.1}$$

We can define a group element in total analogy to the preceding cases

$$g=\exp(x^\mu\,P_\mu)\,\exp(h_a{}^b\,K^a{}_b)\,g_A\equiv g_x\,g_h\,g_A\,,$$

where now

$$\begin{aligned}g_A=\exp\left(\frac{1}{8!}A_{a_1\dots a_8}R^{a_1\dots a_8}\right)\exp\left(\frac{1}{7!}\sum_k\,A^{(k)a_1\dots a_7}R^{(k)}_{a_1\dots a_7}\right)\exp\left(\frac{1}{6!}A_{a_1\dots a_6}R^{a_1\dots a_6}\right)\times\\ \times\exp\left(\frac{1}{2}A_{a_1a_2}R^{a_1a_2}\right)\exp\left(\sum_k\,A^{(k)}_aR^{(k)a}\right)\exp(AR)\,.\end{aligned}$$

We take the commutation relations

$$\begin{aligned}[]
[K^a{}_b,\,K^c{}_d]&=\delta^c_bK^a{}_d-\delta^a_dK^c{}_b\,, &\qquad [K^a{}_b,\,P_c]&=-\delta^a_cP_b\,,\\
[K^a{}_b,\,R^{a_1\dots a_p}]&=\delta^{a_1}_bR^{aa_2}+\dots\,, &\qquad [R,\,R^{a_1\dots a_p}]&=c_pR^{a_1\dots a_p}\,,\\
[R^{a_1a_2},\,R^{b_1\dots b_6}]&=c_{2,6}R^{a_1a_2b_1\dots b_6}\,,
\end{aligned} \tag{4.2}$$

where the index (k) which turns up in the second line for some generators is suppressed as it is unaffected by the action of $K^a{}_b$, as well as the commutation relations

$$\left[R^{(k)a},\, R^{(l)b}\right] = c_{1,1}\delta^{k,l}\, R^{ab}\,, \qquad \left[R^{(k)a},\, R^{a_1\ldots a_6}\right] = c_{1,6}R^{(k)a_1\ldots a_7}$$

and

$$\left[R^{(k)a},\, R^{(l)a_1\ldots a_7}\right] = -c_{1,7}\delta^{k,l}\, R^{aa_1\ldots a_7}. \tag{4.3}$$

The constants are as in (3.2), and $\delta^{(kl)}$ is equal to one if $k = l$ and zero otherwise. The algebra defined by the commutation relations in (4.2), (4.3) we call $G_{IA(n)}$, where the subscript reminds us of the fact that we are still dealing with $N = 1$ abelian theories, however, this time with an arbitrary number of vectors n.

Proceeding as usually by calculating the covariant forms of the non-linear realisation and taking only those that are also covariant under the conformal group we derive the following field strengths which are very similar to those of ten dimensional supergravity with one vector multiplet, just enhanced by an additional index:

$$\tilde{F}_a = \tilde{D}_a A\,, \tag{4.4}$$
$$\tilde{F}^{(k)}_{a_1a_2} = 2e^{-1/4A}\tilde{D}_{[a_1}A^{(k)}_{a_2]}, \tag{4.5}$$
$$\tilde{F}_{a_1a_2a_3} = 2e^{-1/2A}\left(\tilde{D}_{[a_1}A_{a_2a_3]} - 2A^{(k)}_{[a_1}\tilde{D}_{a_2}A^{(k)}_{a_3]}\right) \tag{4.6}$$

for the original fields, and for the duals of those fields we find

$$\begin{aligned}\tilde{F}_{a_1\ldots a_7} &= 7e^{1/2A}\tilde{D}_{[a_1}A_{a_2\ldots a_7]}\,,\\ \tilde{F}^{(k)}_{a_1\ldots a_8} &= 8e^{1/4A}\left(\tilde{D}_{[a_1}A^{(k)}_{a_2\ldots a_8]} - 7A^{(k)}_{[a_1}\tilde{D}_{a_2}A_{a_3\ldots a_8]}\right)\end{aligned}$$

and

$$\tilde{F}_{a_1\ldots a_9} = 9\left(\tilde{D}_{[a_1}A_{a_2\ldots a_9]} - 2\cdot 7A_{[a_1a_2}\tilde{D}_{a_3}A_{a_4\ldots a_9]} + 4A^{(k)}_{[a_1}\tilde{D}_{a_2}A^{(k)}_{a_3\ldots a_9]}\right),$$

where a sum over the (k) indices is understood. Although the addition of vectors changes the theory considerably (anomalies), in terms of field strengths and field equations the change is marginal; the only covariant field equations are similarly to (3.3) given by

$$\tilde{F}^{a_1\ldots a_p} = \frac{1}{(10-p)!}\epsilon^{a_1\ldots a_{10}}\tilde{F}_{a_{p+1}\ldots a_{10}}, \quad \text{for } p = 1, 3\,.$$

and

$$\tilde{F}^{(k)a_1a_2} = \frac{1}{8!}\epsilon^{a_1\ldots a_{10}}\tilde{F}^{(k)}_{a_3\ldots a_{10}}\,. \tag{4.7}$$

These field equations are the correct field equations for the bosonic sector of $N = 1$ supergravity coupled to n abelian vector multiplets.

We now consider how the above algebra may be embedded into a known Kac-Moody algebra. Again we omit the momentum generators and begin by splitting the other generators into three different classes:

$$G^+_{IA(n)} = \left\{K^a{}_b,\, a < b,\, a, b = 1, \ldots, 10\,, \qquad R^{(k)a},\, R^{a_1a_2},\, R^{a_1\ldots a_6},\, R^{(k)a_1\ldots a_7},\, R^{a_1\ldots a_8}\right\},$$

and

$$G^0_{IA(n)} = \left\{ H_a = K^a{}_a - K^{a+1}{}_{a+1},\, a = 1, \ldots, 9,\, D = \sum_a K^a{}_a, \quad R \right\} \tag{4.8}$$

plus $K^a{}_b$ where $a > b$. The elements of $G^+_{IA(n)}$ can be generated by multiple commutators of the generators

$$K^a{}_{a+1} \quad \text{for} \quad a = 1, \ldots, 9\,, \qquad R^{(k)10},\; k = 1, \ldots, n, \qquad \text{and } R^{56789\,10}\,. \tag{4.9}$$

In the cases previously considered by the authors we have identified the generators analogous to those in equation (4.9) as the simple root generators and those in equation (4.8) as elements of the Cartan sub-algebra of the proposed Kac-Moody algebra. However, unlike in the cases previously considered, the number of elements in equations (4.9) and (4.8) does not match except if $n = 1$, which was considered in the last section, and so this identification can not be quite right if $n > 1$.

The resolution of this dilemma is that for $n > 1$ we are not dealing with the maximally non-compact real form of an algebra. Thus the local sub-algebra -the maximally compact sub-algebra- is different. In the non-split case, it too contains elements of the Cartan sub-algebra. We illustrate the point by considering the algebra SO$(8,\, 8+n)$ whose maximal compact sub-algebra is SO$(8) \otimes$ SO$(8+n)$. It is useful to decompose SO$(8,\, 8+n)$ into representations of SL$(8) \otimes$ SO(n). The commutators of the resulting generators are as follows:

$$\begin{aligned}
&[K^a{}_b,\, K^c{}_d] = \delta^c_b\, K_d - \delta^a_d\, K^c{}_d\,, \quad [K^a{}_b,\, R^{(l)c}] = \delta^c_b\, R^{(l)a}\,, \quad [K^a{}_b,\, R^{cd}] = \delta^c_b\, R^{ad} + \delta^d_b R^{ca}\,,\\
&[R^{(l)a},\, R^{bc}] = 0\,, \qquad [R^{(k)a},\, R^b_{(l)}] = -\delta^k_l\, R^{ab}\,, \qquad [R^{(k)}_a,\, R_{(l)b}] = \delta^k_l\, R_{ab}\,,\\
&[K^a{}_b,\, R^{(l)}{}_c] = -\delta^a_c\, R^{(l)}{}_b\,, \qquad [K^a{}_b,\, R_{cd}] = -\delta^a_c\, R_{bd} - \delta^a_d R_{cb}\,,\\
&[S^{kl},\, R^{(m)a}] = \delta^{(km)}\, R^{(l)a} - \delta^{(lm)}\, R^{(k)a}\,, \qquad [S^{kl},\, R^{ab}] = 0 = [K^a{}_b,\, S^{kl}]\,,\\
&[R^{ab},\, R^{(k)}_c] = \delta^{[a}_c\, R^{(k)b]}\,, \qquad [R_{ab},\, R^{(k)c}] = \delta^c_{[a}\, R^{(k)}_{b]}\,,\\
&[R^{(k)a},\, R^{(l)}_b] = \delta^{(kl)} K^a{}_b + \delta^a_b S^{kl}\,, \qquad [R^{ab},\, R_{cd}] = \delta^{[a}_{[c}\, K^{b]}{}_{d]}\,,
\end{aligned} \tag{4.10}$$

where the indices take the values $a, b, c = 1, \ldots, 8$ and $k, l, m = 1, \ldots, n$), and $\delta^{(kl)} = 1$ if $k = l$ and zero otherwise. The position of the (k) index distinguishes between vector and covector with respect to the SO(n) subalgebra. They are simply transposed to each other and thus their entries coincide $R^{(l)a} = R^a_{(l)}$ and $\delta^{(kl)} = \delta^k_l$.

The generators of the maximal compact sub-algebra among (4.10) are given by the anti-symmetric part in $2K_{[ab]} = J_{ab}$ which generate SO(8), and the linear combinations $R_{ab} + R^{ab}$, $R^{(k)}_a - R^{(k)a}$ and S^{kl} which can easily be shown to generate SO$(8+n)$.

Taking the maximal compact subalgebra, i.e. SO(8)$\otimes$SO$(8+n)$, as the local sub-algebra in the non-linear realisation, the coset is of the form $g_{na} g_r$ and contains the generators $K^a{}_b, a \le b$, R^{ab}, $R^{(k)a}$. We divide the coset generators into a commuting set given by $K^a{}_a - K^{a+1}{}_{a+1}$, $a = 1, \ldots, 7$ as well as a trace term $D = \sum_1^8 K^a{}_a$, and the remainder, which can be expressed as multiple commutators of $K^a{}_{a+1}$, $a = 1, \ldots, 7$ and $R^{(k)10}$. We note that the commuting set contains only 8 generators while the remainder is generated by $7+n$ elements, where n is the number of added vectors and thus $n > 1$ in this chapter. Hence we have a mismatch identical to that found above. Indeed, as explained there some of the Cartan sub-algebra generators of the original algebra appear in the subgroup and indeed all the generators S^{kl} are in the sub-algebra.

Having understood the consequences of taking a real form that is not the maximally non-compact real form, we now continue with identifying the Kac-Moody algebra. Let us start by considering the case of only two vector multiplets, i.e. $n = 2$. This does not effect the counting in (4.8), so we still find eleven commuting elements in the coset. However, we find now (4.9) that 12 elements are needed to generate all generators of the non-linear realisation in (4.1) via repeated commutation relations. To identify the underlying algebra, we again restrict the space-time indices to take only seven values $4, \ldots, 10$, and define

$$\begin{aligned} \hat{K}^{-1}{}_i &= \frac{1}{6!}\epsilon_{ii_1\ldots i_6} R^{i_1\ldots i_6}\,, \\ \hat{R}^{(k)-1} &= \frac{1}{\sqrt{2}\,7!}\epsilon_{i_1\ldots i_7} R^{(k)i_1\ldots i_7}, \quad k = 1, 2 \\ \hat{R}^{-1\,i} &= \frac{1}{7!}\epsilon_{i_1\ldots i_7} R^{i_1\ldots i_7, i}\,. \end{aligned} \tag{4.11}$$

In the first line, the six-form generator was dualised and used to define a $\hat{K}^{-1}{}_i$ (index enhancement by one), in the second line the duals of the gauge fields were put into the original gauge field generator (we again obtain an index enhancement, this time for the vectors). In the last equation, the generator corresponds to a field dual of gravity. This generator can be obtained from altering the following commutators in $G_{IA(n)}$

$$\begin{aligned} \left[R^{a_1\ldots a_6}, R^{b_1 b_2}\right] &= -c_{2,6} R^{a_1\ldots a_6 b_1 b_2} + \frac{2}{7} R^{a_1\ldots a_6[b_1, b_2]}\,, \\ \left[R^{(l)a_1\ldots a_7}, R^{(k)b}\right] &= -c_{2,6}\delta^{(kl)}\, R^{a_1\ldots a_6 b_1 b_2} + \frac{2}{7}\sum \delta^{(kl)} R^{a_1\ldots a_7, b}\,. \end{aligned} \tag{4.12}$$

These commutation relations fulfill the Jacobi identities, and they coincide with the commutation relations of the generators of the subalgebra of (4.10) for the case $n = 2$, i.e. SO$(8, 10)$ which are not in the maximally compact sub-algebra. We do not expect to find the generators of the compact sub-algebra as these do not lead to fields in the non-linear realisation since they belong to the local symmetries. Deploying similar arguments to those given in the previous chapters we can argue that the underlying Kac-Moody algebra which allows for the full index range $a, b = 1, \ldots, 10$ must be very-extended D_9 whose real form is very extended SO$(8, 10)$.

We now identify the Cartan sub-algebra and positive simple roots generators of very-extended SO$(8, 8+n)$. We have just noted that when the group involved is not split some of the generators of the Cartan sub-algebra are compact and some are non-compact. Clearly, the compact generators of the Cartan sub-algebra belong to the compact subalgebra and so do not appear explicitly in the non-linear realisation. Using elements of the Cartan sub-algebra of the compact subalgebra SO(n) generated by S^{kl}, one can easily check that for each value of n the number of Cartan generators from this compact subalgebra accounts precisely for the mismatch between maximally commuting non-compact generators (4.8) and the generators that span the positive roots of the coset elements (4.9).

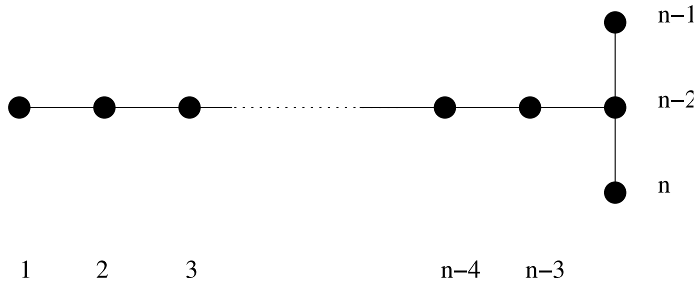

Figure 4: The Dynkin diagram of the D-series.

All Cartan generators of the full very-extended algebra in the case of two added vectors, i.e. an SO(2) subgroup with generator iS^{12}, are uniquely given by

$$\begin{aligned} H_a &= K^a{}_a - K^{a+1}{}_{a+1} \quad \text{for} \quad a = 1, \ldots, 9\,, \qquad H_{10} = K^{10}{}_{10} + iS^{12} + \frac{1}{8}D - \frac{1}{2}R\,, \\ H_{11} &= K^{10}{}_{10} - iS^{12} - \frac{1}{8}D + \frac{1}{2}R\,, \quad H_{-1} = -K^1{}_1 - K^2{}_2 - K^3{}_3 - K^4{}_4 + \frac{1}{4}D - R\,. \end{aligned} \tag{4.13}$$

We note that the linear combination defined in chapter 1 (2.12) and two (3.6) by $K^{-1}{}_{-1} = \frac{1}{8}D - \frac{1}{2}R$ can again be made use of in order to simplify the expressions for H_{10}, H_{11}, and H_{-1}.

We also want to define the simple positive roots which together with (4.13) can be used to derive the complete Dynkin diagram of the Kac-Moody algebra that is called very-extended D_9. They have to meet two requirements: they have to be eigenvectors of the Cartan sub-algebra elements (with respect to the adjoint representation), and they have to be linear combinations of the simple elements found in (4.9). As such they can only be

$$\begin{aligned} E_a &= K^a{}_{a+1} \quad \text{for} \quad a = 1, \ldots, 9\,, \qquad E_{-1} = R^{56789\,10} \\ E_{10} &= R^{(1)10} + iR^{(2)10}\,, \qquad E_{11} = R^{(1)10} - iR^{(2)10}\,. \end{aligned}$$

The occurrence of the imaginary unit is related to the fact that we are in this case dealing with non-split forms of real groups.

We will show next how the above two vector generators correspond to new nodes in the resulting Dynkin diagram. We restrict our attention to the finite Lie sub-algebra the extension to the very extended algebra being obvious. Take a look at figure 4.

By this way of drawing the Dynkin diagram it is understood, that D_n is decomposed into representations of A_{n-2} (the horizontal line) rather than A_{n-1}. Decomposition into representations of A_{n-2} gives 2 vector generators, apart from the known 2-form generator. The decomposition of $\mathrm{SO}(n-1, n+1)$ into representations of $\mathrm{SL}(n-1) \times \mathrm{SO}(2)$ splits the $n(2n-1)$ degrees of freedom of the former into representations $K^a{}_b$, D, R^{ab} and R_{ab}, $R^{a(i)}$ and $R_{a(i)}$ (where $a, b = 1, \ldots, n-1$), plus the SO(2) generator $iS^1{}_2$. The two added vectors effectively span the new direction (called 11 in figure 5). The resulting symmetry algebra is thus D_9, in the real form $\mathrm{SO}(8, 10)$.

The generalisation to an arbitrary number n of abelian vectors should follow the same pattern since the above discussion was not special to the case of only two added vectors. In particular, one can easily allow for arbitrary k in the second line of equation (4.11) where

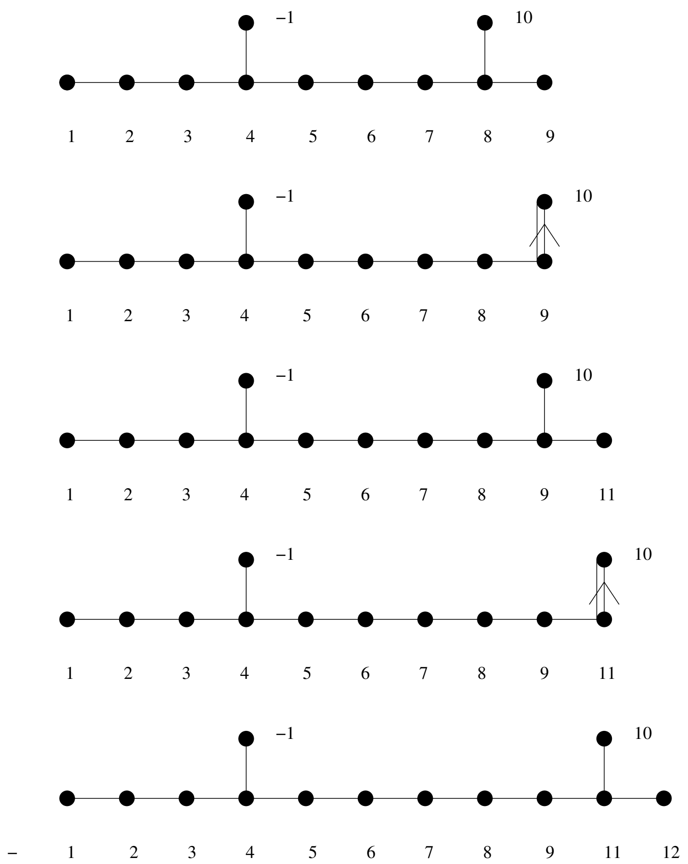

Figure 5: Growing Dynkin diagrams for added vectors.

we restricted the index range to seven space-time dimensions. Every pair (l) of vectors ($E_{8+2l} = R^{(2l-1)\,10} + iR^{(2l)\,10}$ and $E_{9+2l} = R^{(2l-1)\,10} - iR^{(2l)\,10}$ for $l = 1, 2, 3, \ldots$) will just increase the rank of the algebra. If the number of vectors is even ($n = 2k$), then the resulting algebra will be D_{8+k} in the real form $\mathrm{SO}(8, 8+n)$, if the number of vectors is odd ($n = 2k+1$), then the resulting algebra will be B_{8+k} in the real form $\mathrm{SO}(8, 8+n)$, where the last vector stays without partner. The algebra that mixes the different flavours of the abelian vectors is $\mathrm{SO}(n)$. The addition of vectors follows the embedding

$$D_8 \subset B_8 \subset D_9 \subset B_9 \subset \ldots ,$$

where the leftmost group corresponds to pure supergravity from chapter 1, the second entry to one vector as described in chapter 2. This embedding also holds for the very-extended versions of those groups. In terms of Dynkin diagrams the extension works like in figure 5. We have called the extra dimension that showed up after adding the first two vectors "11", and the extra dimension for 4 added vectors, we have called "12" in the above figure 5.

We notice that adding 32 vectors to the theory will give the Kac-Moody algebra of very-extended D_{24}. This algebra was also found to be the symmetry algebra of the bosonic string [11]. Interestingly, the new dimensions in our way of labeling would be called $11, \ldots, 26$. Hence, we find the same Dynkin diagram for a 10-dimensional theory if we add in 32 vectors! However, the Kac-Moody algebra D_{24}^{+++} for the effective bosonic string uses a different real form which for the finite dimensional sub-algebra is $O(24,24)$. In the heterotic case, we required the real form $O(8,40)$. Although the Dynkin diagrams of different real forms coincide because of the same complex algebra they are derived from, we require different real symmetry groups to obtain the correct field content and Lorentz symmetry for each theory. Furthermore, as we have considered only abelian vectors, the SO(32)-symmetry that we find is not a gauge symmetry, and there are no gauge connection terms in the field strengths $F^k_{a_1a_2}$, as can be seen from equation (4.5). Nonetheless, this result goes in the direction of the conjecture that all string theories can be derived from the closed bosonic string [18].

We close this section with some comments on how one might gauge the SO(32) symmetry. Since gravity reduced on a sphere leads to non-abelian gauge symmetries and gravity can be described in an arbitrary background by a non-linear realisation whose underlying Kac-Moody algebra is thought to be A_{D-3}^{+++} [19] it must be possible to include non-abelian vector fields in the framework developed in this paper. Indeed, there is a close analogy between the way the dual fields for gravity and the dual gauge fields occur in the Kac-Moody algebra. The corresponding generators are $R^{a_1\ldots a_7,b}$ and $R^{(l)a_1\ldots a_7}$ respectively and these generators both first occur when considering the restriction of the the index range to take seven values $(4,\ldots,10)$ since their index structure is fairly similar.

We note that if we allow the derivatives of the gauge fields to become covariant derivatives then the equation

$$D_aF^{(k)ab} = \frac{1}{2!7!}\epsilon^{ba_1\ldots a_9}\, H_{a_1\ldots a_7}\, F^{(k)}_{a_8a_9}\,.$$

is indeed the correct field equation for the gauge vectors of heterotic supergravity in 10 dimensions. It is just the non-abelian analogue of the abelian field equation given in (4.7).

5. Discussion

In this paper we have found that the bosonic sector of the $N=1$ supergravity theories in ten dimensions coupled to any number n of abelian multiplets can be described by a non-linear realisation and we have argued that this theory can be extended in such a way that it possess a Kac-Moody algebra that is $D_{8+\frac{n}{2}}$ if n is even and $B_{8+\frac{n-1}{2}}$ if n is odd. However, the construction uses particular real forms which for the finite dimensional sub-algebras are $O(8,8+n)$.

These underlying Kac-Moody algebras are consistent with a number of related results in the literature. It has been known [20] for many years that the supergravity system mentioned above when dimensionally reduced to d, $d\geq 4$ space-time dimensions possesses scalars which are in a non-linear realisation of the algebra $O(d,d+n)$ with local subgroup $O(d)\times O(d+n)$. The symmetry algebra in lower dimensions can be read off from the Kac-Moody Dynkin diagram of the theory in ten dimensions by deleting nodes from the left. Carrying out this procedure, we indeed find the symmetry algebra of reference [20].

It has been found that the dynamics of certain theories of gravity coupled to a dilaton and n-forms near a space-like singularity becomes a one dimensional motion with scattering taking place in the Weyl chambers of certain overextended Lie algebras [21]. This motion is sometimes referred to as cosmological billiards. For $N = 1$ pure supergravity theory the corresponding algebra that appears is over-extended D_8 (denoted D_8^{++}) and for this theory extended with a vector field is over-extended B_8 (denoted B_8^{++}) [21].

It has also been shown [22] that the T-duality transformations for the IIA string theory in ten dimensions reduced on an k-torus for $k = 1, \ldots, 10$ have a natural action on the moduli space of the k-torus that is the Weyl group of E_k. The corresponding result for the supergravity with 32 vector fields reduced to one dimensions is the Weyl group of over-extended O(8,24) [23].

The existence of these huge symmetry groups in gravity theories must be related to the symmetries of solutions to these theories, and might thus be utilised to find solutions of the supergravity theories [24]. More importantly, it has been suggested [10]–[16] and [19, 25, 26] that the non-linear realisation constructed from any very-extended algebra G^{+++}, denoted $\mathcal{V}_G$ in [25], can provide a consistent theory which extends that of gravity. The results of this paper provided evidence for this idea and indeed have already played a role in some of the just mentioned papers.[1] It also encourages the conjecture of reference [25] namely that all these theories $\mathcal{V}_G$ might be contained as special limits of a single theory.

In this paper we constructed the respective theories as non-linear realisations and then deduce the underlying Kac-Moody algebra that could be a symmetry of the extended theory. However, given the Kac-Moody algebra one can reverse the process and construct its non-linear realisation at low levels. This is essentially the same calculation, but read in reverse. Indeed, one can check that the generators in D_8^{+++} at low levels, which were given in reference [25], are exactly those required to give rise to the fields of $N = 1$ supergravity and that the corresponding group element is that of equation (2.3).

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